The constraints as evolution equations for numerical relativity

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Abstract. The Einstein equations have proven surprisingly difficult to solve numerically. A standard diagnostic of the problems which plague the field is the failure of computational schemes to satisfy the constraints, which are known to be mathematically conserved by the evolution equations. We describe a new approach to rewriting the constraints as first-order evolution equations, thereby guaranteeing that they are satisfied to a chosen accuracy by any discretization scheme. This introduces a set of four subsidiary constraints which are far simpler than the standard constraint equations, and which should be more easily conserved in computational applications. We explore the manner in which the momentum constraints are already incorporated in several existing formulations of the Einstein equations, and demonstrate the ease with which our new constraint-conserving approach can be incorporated into these schemes.

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1. Introduction

Despite significant recent advances in both computational power and algorithmic complexity, there remain significant unresolved problems with numerical implementations of the Einstein equations. Perhaps the most exciting recent developments are the many new three-plus-one dimensional formulations of these equations, which, at least in part, provide greater stability than the original ADM formulation (see [1] for a review).

In particular, formulations of the Einstein equations in strongly hyperbolic, flux-conservative form have opened the way for the application of algorithms and techniques originally developed for computational fluid dynamics (see, for example, [4]). Despite the great promise and significant advances, modern numerical relativity codes are still unable to fully simulate the complete coalescence of binary black hole systems. A full understanding and complete simulation of such systems are of vital importance to the analysis of gravitational wave signals collected by LIGO and similar gravitational wave detectors.

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Until recently, almost all three-plus-one dimensional formulations of the Einstein equations have shared one common feature — the constraint equations are monitored through evolution, but never again solved once the initial data is constructed. An exception to this is the Texas group, who resolve the elliptic constraint equations after every few timesteps [2]. The four constraints are more typically treated as additional conditions which are monitored while the spacetime is evolved. In fact the constraints, and the Hamiltonian constraint in particular, have been found to be excellent prognosticators for the accuracy and stability of the numerical solution. The rule of thumb appears to be that when the Hamiltonian constraint "explodes", the code will crash shortly thereafter. To the best of our knowledge, all three-plus-one dimensional formulations of the Einstein equations suffer from these fundamental problems.

In this paper we propose an approach to transforming all four constraint equations into evolution equations for a new set of dynamical variables — essentially the conformal factor and it's spatial derivatives, quantities which reduce to the Newtonian potential and force in the weak-field limit. Our formulation is based on the standard York-Lichnerowicz conformal decomposition and split of the extrinsic curvature into its trace and trace-free parts. In this sense it is similar in spirit, if not in detail, to the standard approach to the initial value problem [8] and several modern formulations of the evolution equations [9, 10].

The underlying motivation for the new formulation presented in this paper is the belief that it is the violation of the constraint equations, and resultant generation of spurious energy-momentum sources, which lead to instabilities in numerical implementations of Einstein's equations. By solving the Hamiltonian and momentum constraints directly, especially in regions of the domain where the gravitational fields are strongly dynamic, we automatically prevent the run-away errors which typically appear in the Hamiltonian constraint. By automatically satisfying the constraints, we guarantee conservation of energy-momentum.

The formulation of the constraints described in this paper is closest to that of Bona, Massó *et al* [3, 4, 5, 6], who use the momentum constraints as evolution equations. In their early papers the constraints were used purely to eliminate spatial derivatives of the extrinsic curvature, thereby casting the equations in strongly hyperbolic form [4]. Only later are the momentum constraints explicitly described as evolution equations [6], although they had always played that role in the formulation. More recently Bona *et al* [7] have incorporated the constraints into the evolution system by expanding the Einstein equations themselves, thereby maintaining general covariance. Our current approach follows the spirit of their earlier, non-covariant formulations.

We show that the BSSN formulation [9, 10] shares important features with the Bona-Massó (BM) approach, in particular their common treatment of the momentum constraints. We demonstrate that the BSSN equations incorporate the momentum constraints as evolution equations, and we highlight similarities in the way the momentum constraints are treated in the BM and BSSN approaches. We then develop our own formulation of the momentum constraints, as well as a new approach to rewriting the Hamiltonian constraint as an evolution equation. The goal of this formulation is the automatic conservation of energy-momentum which we achieve by directly solving the constraint equations, thus guaranteeing that they are satisfied throughout the evolution.

We proceed as follows. In the next section we briefly discuss the constrained

nature of general relativity, followed by an outline of the approach taken by Bona and Massó in treating the momentum constraints. Turning to the BSSN approach, we highlight the momentum-conserving properties of the algorithm, before developing our own approach. We rewrite both the Hamiltonian and momentum constraints as evolution equations, and demonstrate how transparently these equations can be incorporated into existing evolution algorithms. Finally, we consider both the advantages and the issues raised by our results.

2. Constraints in General Relativity

The constraint equations in Einstein's 1915 geometric theory of gravitation play a cornerstone role – general relativity is a fully constrained theory. If the four constraints are satisfied at every point over every possible spacelike hypersurface of a spacetime, then, necessarily, the entire spacetime is a solution of all Einstein's equations. It is surprising that the constraints have not played a more pivotal role in numerical relativity, given their central importance in theoretical developments of the theory.

The four constraint equations per point in spacetime are

$$\mathcal{H} \;= R + (\mathrm{Tr}\,K)^2 - K_{ij}K^{ij} - 2\rho = 0 \tag{1}$$

$$\mathcal{H}_i = \nabla_j K^j{}_i - \nabla_i \,\mathrm{Tr}\,K - S_i = 0, \tag{2}$$

known as the Hamiltonian and momentum constraints, respectively. The extrinsic curvature K_{ij} is defined as

$$K_{ij} = -\frac{1}{2N}\left\{\partial_t g_{ij} - 2\nabla_{(i}N_{j)}\right\}, \tag{3}$$

with ρ and S_i representing the energy-momentum source terms. In the remainder of this paper we work in vacuum ($\rho = S_i = 0$), although our analysis is equally valid in the presence of matter.

As it stands, these constraints relate the six components each of g_{ij} and K_{ij}, and represent an initial value problem which must be solved to obtain consistent initial data. Mathematically the constraints are conserved by the evolution equations, implying that if they are satisfied on one slice of a foliation, they will be satisfied on all future slices. However, it is well know that computational implementations suffer from serious errors which propagate in one or more of the constraints. It is this problem which we hope to overcome by recasting the constraints as evolution equations.

3. Bona-Massó treatment of the momentum constraints

It is our goal to recast all four constraint equations as evolution equations by introducing a set of new variables. The goal is to expand the system of evolution equations to include the traditional constraints, which necessarily introduces a set of new "subsidiary constraints" which must be monitored during evolution. These subsidiary constraints take the place of the original Hamiltonian and momentum constraints.

Such an approach has already been taken, perhaps indirectly, by Bona and Massó *et al* while developing hyperbolic formulations of the evolution equations [3, 4, 5, 6]. Using the momentum constraints to ensure the hyperbolicity of the full set of first-order, flux-conservative evolution equations [4], the Bona-Massó (BM) formulation

introduces a new set of variables

$$V_i = \frac{1}{2}\, g^{jk} \left(\partial_i g_{jk} - \partial_j g_{ik}\right). \tag{4}$$

Evolution equations for the V_i can be obtained by differentiation and simplification using the $\dot{g}_{ij}$ equation together with the momentum constraints [4]. However, the same result can be obtained directly from the momentum constraints themselves, which take the form [6]:

$$\partial_t V_k = -2K^{ij} \left(d_{ijk} - \delta_{ik} d^l{}_{lj}\right) + K^{ij} \left(d_{kij} - \delta_{ik} d_{jl}{}^l\right) \tag{5}$$

where we have taken $N = 1, N^i = 0$ for brevity, and

$$d_{kij} = \frac{1}{2}\, \partial_k g_{ij} \tag{6}$$

are the first-order flux variables required to reduce the spatial order of the Einstein equations. In this approach the definition of V_i, equation (4), is relegated to the status of a constraint relating the independent variables V_k, g_{ij} and d_{kij}. These are the subsidiary constraints in the Bona-Massó approach.

Bona and Massó provide an elegant incorporation of the momentum constraints into the set of dynamical equations, although the four new subsidiary constraints (4) are still of a moderately complex form, with no obvious manner of enforcing them naturally within an evolution scheme. Before describing an approach to incorporating the Hamiltonian constraint into the evolution equations, we pause to consider the status of the constraint equations in the BSSN formulation.

4. Momentum conservation in the BSSN formulation

As outlined above, Bona and Massó have shown that the momentum constraints can be rewritten as first-order evolution equations, which take the form (5) when $N = 1$ and $N^i = 0$. They introduce the new variable V_i, defined by equation (4), which appears naturally in the momentum constraints when one commutes the spatial derivatives of K_{ij} with the time derivatives of g_{ij} implicit in the definition of extrinsic curvature. This at once removes troublesome spatial derivatives, and explains why eliminating spatial derivatives in the evolution equation for V_i with the momentum constraints is in fact equivalent to using the momentum constraints themselves as evolution equations.

In apparent contrast, the BSSN formulation introduces the new variables

$$\tilde{\Gamma}^i = -\partial_j \tilde{g}^{ij}, \tag{7}$$

where $\tilde{g}^{ij}$ is the contravariant conformal metric. An evolution equation is then obtained by differentiation and commutation of spatial and temporal derivatives. However, the BSSN formulation uses precisely the same approach to eliminating spatial derivatives of the conformal, trace-free portion of the extrinsic curvature $\tilde{A}^{ij}$ as described by Bona and Massó [4]. In doing so, the BSSN evolution equation for $\tilde{\Gamma}^i$ becomes equivalent to the evolution equation obtained directly from the momentum constraints themselves, in precisely the same manner as the BM formulation.

In fact, the BM variable V_i and the BSSN $\tilde{\Gamma}^i$ are closely related. Under the conformal decomposition

$$g_{ij} = e^{4\phi}\, \tilde{g}_{ij} \qquad \text{with} \qquad \det(\tilde{g}_{ij}) = 1 \tag{8}$$

the BM variables become

$$V_i = \tilde{V}_i + 4\, \partial_i \phi \tag{9}$$

where $\tilde{V}_i$ is defined like V_i, but in terms of the conformal metric $\tilde{g}_{ij}$. We see that V_i splits naturally into a portion $\tilde{V}_k$ determined entirely by the conformal geometry, and the remainder which depends on the scale factor. A similar relationship was obtained previously in the case of a static conformal factor [5].

Expanding the conformal portion of V_i, and using the constraint $\det \tilde{g} = 1$, we find that

$$\tilde{V}_j = -\frac{1}{2}\,\tilde{g}_{ij}\,\tilde{\Gamma}^i, \tag{10}$$

clearly showing that the BSSN variable $\tilde{\Gamma}^i$ is just the conformal part of the BM variable V^i. With this realization, it is straightforward to use the momentum constraints, in the form of equation (5), to obtain an evolution equation for $\tilde{\Gamma}^i$. Not surprisingly, the resulting equation is precisely the one used in the standard BSSN formulation to evolve $\tilde{\Gamma}^i$.

In their extensive analysis and review of existing formulations of the Einstein equations, Shinkai and Yoneda note that the advantages of the BSSN system over the standard ADM formulation are due entirely to the introduction of the $\tilde{\Gamma}^i$ variables, and the subsequent elimination of spatial derivatives using the momentum constraint [11]. In other words, the sole advantage of the BSSN approach is the *use of the momentum constraints as evolution equations*. The BSSN formulation uses the momentum constraints to evolve the conformal portion of V_i, and introduces the constraints (7) in their place.

The relative success of the BSSN formulation provides strong motivation for incorporating the Hamiltonian constraint into the set of evolution equations. We do this below, as well as proposing an alternative formulation of the momentum constraints, with the advantage of a set of extremely natural, and very simple, subsidiary constraints. As we shall see, our approach to the momentum constraints is opposite to the BSSN choice, since we evolve that portion of V_i arising directly from the conformal factor.

5. Energy conservation: the Hamiltonian constraint as evolution equation

The key to rewriting the Hamiltonian constraint as an evolution equation is performing a conformal decomposition on the three-metric. Rewriting the extrinsic curvature in terms of its trace and trace-free parts will allow us to use the Hamiltonian constraint to evolve the scale factor.

We begin with a conformal decomposition of the three-metric,

$$g_{ij} = e^{4\phi}\,\tilde{g}_{ij} \tag{11}$$

with $\det \tilde{g}_{ij} = 1$, and split the extrinsic curvature into its trace and trace-free parts,

$$K_{ij} = A_{ij} + \frac{1}{3}\,g_{ij}\,\mathrm{Tr}\,K \tag{12}$$

where $\mathrm{Tr}\,A = 0$. This allows us to rewrite the Hamiltonian constraint as

$$\mathcal{H} = R - A_{ij}A^{ij} + \frac{2}{3}\,(\mathrm{Tr}\,K)^2 - 2\rho, \tag{13}$$

where R is the full three-dimensional Ricci scalar calculated from the physical three-metric g_{ij}. It is therefore a function of ϕ and $\tilde{g}_{ij}$, together with their spatial derivatives [8]. It is not a function of $\dot{\phi}$, and thus does not play a vital role in the current analysis.

We could in principle expand R, as York does, into terms containing spatial derivatives of ϕ together with the conformal curvature $R(\tilde{g})$.

Our aim is to obtain an evolution equation for the conformal factor from the Hamiltonian constraint, which requires an understanding of where $\dot{\phi}$ appears in the constraint. Writing

$$A_{ij} = \left(\delta_i^a \, \delta_j^b - \frac{1}{3} \, g_{ij} \, g^{ab} \right) K_{ab} \tag{14}$$

and applying the conformal decomposition to the definition of extrinsic curvature, equation (3), we have

$$K_{ab} = -\frac{1}{2N} \left(e^{4\phi} \partial_t \tilde{g}_{ab} - 2\nabla_{(a} N_{b)} + 4 g_{ab} \dot{\phi} \right) \tag{15}$$

and since

$$\left(\delta_i^a \, \delta_j^b - \frac{1}{3} \, g_{ij} \, g^{ab} \right) g_{ab} = 0, \tag{16}$$

it is clear that A_{ij} does not depend directly on the time development of ϕ. The only functional dependence on $\dot{\phi}$ in the Hamiltonian constraint is thus within the $\mathrm{Tr}\, K$ term.

Under the conformal decomposition (11) the trace of the extrinsic curvature becomes

$$\mathrm{Tr}\, K = \frac{1}{N} \nabla_i N^i - \frac{6\dot{\phi}}{N}, \tag{17}$$

and noting that the Lie derivative of the conformal factor along the shift vector N^i is given by

$$\mathcal{L}_N \phi = N^k \partial_k \phi + \frac{1}{6} \, \partial_k N^k = \frac{1}{6} \nabla_k N^k, \tag{18}$$

the trace of the extrinsic curvature can be written as

$$\mathrm{Tr}\, K = -\frac{6}{N} \left(\partial_t - \mathcal{L}_N \right) \phi. \tag{19}$$

This equation is often used to directly evolve the conformal factor [10]. Instead, we treat it as a definition of $\mathrm{Tr}\, K$ in terms of the time development of the conformal factor, allowing us to rewrite the Hamiltonian constraint as the evolution equation for ϕ.

The time development of the conformal factor only enters the Hamiltonian constraint through the term quadratic in $\mathrm{Tr}\, K$. Proceeding formally, we use equation (19) to rewrite the Hamiltonian constraint $\mathcal{H} = 0$ as

$$\left(\partial_t - \mathcal{L}_N \right) \phi = \pm \frac{N}{6} \sqrt{\frac{3}{2} \left(A_{ij} A^{ij} - R + 2\rho \right)}. \tag{20}$$

This is a problematic result. The square root causes strife whenever those terms within it approach zero (potential computational problems) or become negative. In computational tests we find that errors which typically appear as violations of the Hamiltonian constraint when using the ADM evolution equations do indeed reappear as problems “under the square root”.

There are several ways of proceeding from the Hamiltonian constraint (13). In general there is no need to replace both $\mathrm{Tr}\, K$ terms, since we wish to maintain linearity in the time derivative of the conformal factor. We can therefore write

$$\left(\partial_t - \mathcal{L}_N \right) \phi = \frac{N}{4} \left(\frac{R(g) - A_{ij} A^{ij} - 2\rho}{\mathrm{Tr}\, K} \right), \tag{21}$$

providing an evolution equation for the conformal factor which does not suffer from the problems mentioned above. However, problems can still arise if the denominator is zero.

We note that equation (21) can be rewritten as an evolution equation for a new variable

$$\xi = \frac{\operatorname{Tr} K\, \phi}{N}, \tag{22}$$

with the advantage that the evolution equation itself is singularity free when $\operatorname{Tr} K = 0$. Although this form may be more advantageous in computations, it merely relocates the problem to the calculation of ϕ once ξ is known.

Another alternative is to monitor $\operatorname{Tr} K$ within the code, replacing equation (21) with an alternative whenever the absolute magnitude of $\operatorname{Tr} K$ falls below some threshold. For example, equation (19) implies that

$$(\partial_t - \mathcal{L}_N)\, \phi = 0 \tag{23}$$

whenever the trace of the extrinsic curvature is zero. For the remainder of this paper we will use equation (21).

6. Momentum conservation

We now turn to the momentum constraints, rewritten in terms of the conformal decomposition described in the previous section. Our aim is to find a new set of variables which can be evolved using the momentum constraints, and which have a simpler subsidiary constraint structure than the BM formulation, equation (4).

The momentum constraints are

$$\mathcal{H}_j = \partial_i K^i{}_j + \Gamma^i_{ik} K^k{}_j - \Gamma^k_{ij} K^i{}_k - \partial_j \operatorname{Tr} K \tag{24}$$

into which we can substitute the definition of $K^i{}_j$ expressed in terms of its trace and trace-free parts. Combining the results in the previous section we find that

$$K_{ij} = A_{ij} + \frac{1}{3N}\, g_{ij} \left(\nabla_k N^k - 6\dot{\phi} \right), \tag{25}$$

and thus the momentum constraints are

$$\mathcal{H}_j = \nabla_i A^i{}_j - \frac{2}{3}\partial_j \left(\frac{\nabla_k N^k}{N} \right) + 4\partial_j \left(\frac{\dot{\phi}}{N} \right).$$

The plan is to define the new variable

$$\Phi_j = \partial_j \phi, \tag{26}$$

and commute the temporal and spatial partial derivatives in the momentum constraint to obtain an evolution equation for Φ_j. We note that Φ_j can be viewed as that portion of Bona-Massó's V_i variable which is derived purely from the scale factor, as shown by equation (9). This is opposite to the choice made in the BSSN formulation, where the momentum constraints are used to evolve the conformal portion of V_i.

Continuing in the manner outlined above, the momentum constraints take the form

$$\partial_t \Phi_j = \frac{1}{6}\partial_j \nabla_k N^k - \frac{1}{6}\, \operatorname{Tr} K\, \partial_j N - \frac{N}{4} \nabla_i A^i{}_j, \tag{27}$$

which provides an evolution equation for Φ_j. However, this can be recast in the more convenient form

$$(\partial_t - \mathcal{L}_N)\,\Phi_j = \frac{1}{6}\partial_j\partial_k N^k - \frac{1}{6}\,\mathrm{Tr}\,K\,\partial_j N - \frac{N}{4}\nabla_i A^i{}_j, \tag{28}$$

by expanding the covariant derivative of the shift. This is the desired evolution equation, derived from the momentum constraints, for the new variables Φ_j.

The stability properties of a formulation involving this evolution equation for Φ_j remain to be investigated. The BSSN formulation explicitly removes spatial derivatives of A^{ij} (or its conformal part) to improve stability. However, we argue that it is the fact that this procedure introduces the momentum constraint as an evolution equation which improves the overall stability, not simply the removal of spatial derivatives.

7. Implementing the "constrained evolution" algorithm

In the previous sections we described how the standard constraint equations can be recast as evolution equations for the conformal factor and its spatial derivatives. It is not hard to see that these new forms of the constraints can be easily "bolted onto" existing formulations of the evolution equations.

We first consider the classic $\dot{g}$-$\dot{K}$, or ADM, formulation of the Einstein equations [12]. Incorporating equations (21) and (28) into the ADM system,

$$(\partial_t - \mathcal{L}_N)\,g_{ij} \;=\; -\,2NK_{ij} \tag{29}$$

$$(\partial_t - \mathcal{L}_N)\,K_{ij} = N\left(R_{ij} - 2K_{il}K^l{}_j + \mathrm{Tr}\,K K_{ij}\right) - \nabla_i\nabla_j N, \tag{30}$$

can be approached in a number of ways. We could, for example, introduce A_{ij}, $\mathrm{Tr}\,K$, ϕ and Φ_j as auxiliary variables, continuing to treat the physical metric and extrinsic curvature as the fundamental variables. This is not the most straightforward approach, and in general, it would seem advantageous to explicitly construct a conformal, trace-free generalization of the ADM equations.

This approach evolves ϕ and $\tilde{g}_{ij}$ in place of g_{ij}, and replaces K_{ij} with $\mathrm{Tr}\,K$ and $\tilde{A}_{ij}$. The resulting equations are identical to the equivalent BSSN equations, namely

$$(\partial_t - \mathcal{L}_N)\,\tilde{g}_{ij} \;=\; -\,2N\tilde{A}_{ij} \tag{31}$$

$$(\partial_t - \mathcal{L}_N)\,\tilde{A}_{ij} = e^{-4\phi}\left[NR_{ij} - \nabla_i\nabla_j N\right]^{\mathrm{TF}} \tag{32}$$

$$\qquad +\; N\left(\mathrm{Tr}\,K\,\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right) \tag{33}$$

where $\tilde{A}_{ij} = e^{-4\phi}A_{ij}$ and "TF" denotes the trace-free portion of the bracketed expression. Various expressions can be derived to evolve $\mathrm{Tr}\,K$, including the standard BSSN equation

$$(\partial_t - \mathcal{L}_N)\,\mathrm{Tr}\,K = N\left\{\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}\left(\mathrm{Tr}\,K\right)^2\right\} - \nabla_i\nabla^i N + 2N\rho. \tag{34}$$

The conformal factor ϕ and its spatial derivatives Φ_j are then evolved using the Hamiltonian and momentum constraints, equations (21) and (28):

$$(\partial_t - \mathcal{L}_N)\,\phi \;=\; \frac{N}{4}\left(\frac{R(g) - A_{ij}A^{ij} - 2\rho}{\mathrm{Tr}\,K}\right) \tag{35}$$

$$(\partial_t - \mathcal{L}_N)\,\Phi_j = \frac{1}{6}\partial_j\partial_k N^k - \frac{1}{6}\,\mathrm{Tr}\,K\,\partial_j N - \frac{N}{4}\nabla_i A^i{}_j. \tag{36}$$

This formulation differs from the existing BSSN approach in two ways. First, the momentum "constraint" is used to evolve the variables Φ_j in place of $\tilde{\Gamma}^i$. Second, the conformal factor is evolved using the Hamiltonian "constraint", rather than being driven directly by the trace of the extrinsic curvature through equation (19).

Similarly, an evolution algorithm based more directly on the BSSN formulation [9, 10] can be obtained. The standard BSSN evolution equations are used to evolve $\tilde{g}_{ij}$, $\tilde{A}_{ij}$ and $\mathrm{Tr}\, K$, as described above, together with the conformal connection functions $\tilde{\Gamma}^i$ [10]. Since these conformal connection functions are already evolved using the momentum constraints in the BSSN formulation, our goal of using all four "constraint equations" as evolution equations can be achieved by incorporating the Hamiltonian equations into the evolution scheme with minor modifications.

Our form of the Hamiltonian constraint, equation (21), is used to replace the standard BSSN evolution equation for the conformal factor ϕ, which is simply equation (19). In this extended-BSSN approach the fundamental variables are still the original set, namely $\phi, \tilde{g}_{ij}, \tilde{A}_{ij}, \mathrm{Tr}\, K$ and $\tilde{\Gamma}^i$, with the only alteration being that the conformal factor is evolved using (21) rather than (19). This fully-constrained approach is a trivial addition to the standard BSSN formulation.

8. Discussion

By recasting the standard Hamiltonian and momentum constraints as evolution equations, we have guaranteed that the evolution scheme conserves energy-momentum. We have also introduced the four subsidiary constraint equations

$$\det \tilde{g}_{ij} = 1 \tag{37}$$

$$\partial_i \phi = \Phi_i \tag{38}$$

in place of the original Hamiltonian and momentum constraints. These subsidiary constraints, or consistency conditions, ensure that the derived variables Φ_j and ϕ, which are evolved independently, continue their expected relation to the original set of physical variables.

The new subsidiary constraints are much simpler than the original Hamiltonian and momentum constraints. Codes which violate the Hamiltonian and momentum constraints introduce, through computational errors, spurious energy-momentum sources. However, violation of the subsidiary constraints does not result in the same physical problem, since the original constraints (now in the form of evolution equations) are still satisfied to machine precision. Thus no spurious energy-momentum can be injected into the system by purely computational errors. It seems likely that this advantage explains much of the success of the BSSN formulation, in comparison with the ADM form of the equations. In this sense, the BSSN approach may be viewed as a natural generalization of the ADM evolution equations to incorporate the momentum constraints.

The major difficulty with the constraint-evolution equations we have obtained is the potential for singularities to develop in the Hamiltonian, arising from zeros in the denominator. Although this problem can be formally avoided by writing the equation in terms of a new variable, the conformal factor must still be calculated from this new variable. As such, this approach merely removes singularities from the evolution equations without wholly avoiding the problem. One solution, at least for black hole spacetimes, would be to use constant-crunch ($\mathrm{Tr}\, K = \text{constant} \neq 0$) slicings, which have been shown to provide potentially stable foliations of black hole spacetimes [13].

This is, however, not an entirely satisfactory solution, and we continue to investigate this issue.

Work is currently under way to investigate the relative stability of this new formulation of the Einstein equations. In particular, we are interested in the relative merits of the various forms of the momentum equations (BSSN $\tilde{\Gamma}^i$, BM V_i, Φ_j). We are also exploring the integration of the full set of constraints into the Bona-Massó and BSSN formalisms to determine their mathematical structure, and performing computational tests of the new constraint equations.

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